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On the plane contact problem of a functionally graded elastic layer loaded by a frictional sliding flat punch†

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Abstract

This article is concerned with the contact mechanics of a functionally graded layer loaded by a frictional sliding flat punch. The coefficient of friction is assumed to be constant and the lower side of the graded layer is firmly attached to a rigid foundation. The graded, nonhomogeneous property of the medium is represented in terms of an exponential variation of the shear modulus, while Poisson's ratio is taken to be constant. Based on the use of plane elasticity equations and the Fourier integral transform technique, the formulation of the current contact mechanics problem lends itself to a Cauchy-type singular integral equation of the second kind for the unknown contact pressure, which is solved numerically. As a result, the effects of several parameters, i.e., the material nonhomogeneity, the friction coefficient, the punch width, and Poisson's ratio, on the distributions of the contact pressure and the in-plane surface stress component are presented.

Keywords**:** Contact mechanics; Flat punch; Functionally graded materials; Nonhomogeneity; Singular integral equation

1. Introduction

Functionally graded materials have been increasingly used in a wide range of modern engineering practices over the past decade, motivated by a number of technological advantages achievable from their attractive features of predetermined, continuous variations of thermophysical properties in the spatial domain [1]. In particular, in conjunction with such tailoring capability of material property gradation that could enhance the surface properties, resolution of various important and interesting issues related to contact responses entailing the graded, nonhomogeneous constituents is posing refreshing challenges. This is because the results of contact mechanics analyses could find broad and direct scientific and industrial applications where the surface wear and damage due to sliding contact are a serious concern, as would be the case in the design of load transfer components or assemblages with the graded properties near and at the contact surfaces [2, 3].

Among others, one of the earlier attempts concerned with the problem of a frictionless rigid punch on a nonhomogeneous half-plane can be attributed to Bakirtas [4]; and rather recently, with the evolution of the functionally graded materials being extended to the field of tribological applications, notable contributions have been made by Suresh and his associates in a series of papers dealing with the contact mechanics of graded media. Specifically, Giannakopoulos and Suresh [5, 6] examined the axisymmetric problem of graded half-spaces subjected to frictionless indenters with different profiles, by considering the elastic modulus that varies with depth either as a power or an exponential function. Finite element and experimental analyses of spherical frictionless indentation of compositionally graded materials were also undertaken by Suresh et al. [7]. Subsequently, Suresh et al. [8] illus-

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trated that the controlled gradients in the mechanical properties of compositions and structures offer unique opportunities for the design of surfaces with improved resistance to sliding-contact deformation and damage that are frequently encountered in the conventional homogeneous substrate [9]. Similar results were observed by Jitcharoen et al. [10] such that the elastic-modulus-graded surfaces could alter the stress field around the punch, thereby suppressing the Hertzian cracking at the edges of the contact region. For a graded semi-infinite substrate possessing the elastic modulus that increases monotonically according to a power-law variation, the closed-form solution to the plane elasticity problem of a sliding rigid cylindrical punch was given by Giannakopoulos and Pallot [11].

Aizikovich et al. [12] studied the Hertzian contact problem for a rigid spherical indenter acting on both the layered and functionally graded half-spaces, and Guler and Erdogan [13, 14] evaluated the contact stress fields for the graded coatings bonded to homogeneous substrates. Guler and Erdogan [15] also examined the problem of contact between two deformable elastic bodies with graded coatings, while El-Borgi et al. [16] discussed the frictionless receding contact of a functionally graded layer pressed against a homogeneous substrate. Ke and Wang [17, 18] utilized a multilayered model-based procedure for the two-dimensional contact analysis of graded coatings with arbitrary material property variations, in which the graded region was simulated as a stack of a number of sublayers with the shear modulus varying linearly in each of the sublayers and continuous at the subinterfaces. Most recently, the multilayered approach was equally applied by Liu et al. [19] in solving the axisymmetric frictionless contact problem of functionally graded materials. With the intention of providing some insights into the design of coatings, Stephens et al. [20] performed a finite element analysis to investigate the initial yielding behavior in a hard coating/substrate system with functionally graded interface under the simplifying assumption of a frictional Hertzian contact pressure profile, indicating the beneficial influence of gradients in yield strength or elastic modulus on the reliability of the coated system. Additional results that address the contact response of graded materials on the basis of the aforementioned Hertzian pressure distribution are to be found in [21-23].

The objective of the present article is to further investigate the problem of contact mechanics for a functionally graded layer loaded by a frictional sliding flat punch. With the friction coefficient being constant, it is assumed that the lower side of the graded layer is fixed to a rigid foundation and the sliding motion of the punch is sufficiently slow to justify the disregarding of inertia effects. The plane elasticity equations are employed in formulating the proposed contact problem and the graded layer is treated as a nonhomogeneous medium. The corresponding shear modulus is expressed in the form of an exponential function varying along the layer thickness and Poisson's ratio is taken to be constant. It now appears to be appropriate to remark that for the solutions to counterpart problems of contact mechanics that involve a homogeneous layer bonded to or resting on a rigid foundation, one can refer to the previous works, for example, in [24-28] and other references quoted therein. Based on the Fourier integral transform method, a Cauchy-type singular integral equation of the second kind is derived for the unknown contact pressure. Numerical results are then provided to demonstrate how the distributions of the contact pressure and the in-plane component of the surface stress are affected by various material, loading, and geometric parameters of the graded layer subjected to the sliding flat punch with friction.

2. Problem statement and basic equations

The problem under consideration is schematically illustrated in Fig. 1, where a functionally graded layer of thickness h is in contact with a rigid flat punch of width $2c$ and is firmly attached to a rigid foundation. The punch is pressed against the layer upper surface by a normal force P and slides slowly in the positive y-direction. A frictional tangential force $\mu_f P$ is developed at the contacting interface by Coulomb's law of friction, with μ_f being the coefficient of friction. The material nonhomogeneity of this graded medium is represented by the shear modulus $\mu(x)$ that follows an exponential variation as

$$\mu(x) = \mu_o e^{\beta x}, \quad \beta = \frac{1}{h}\ln\left(\frac{\mu_h}{\mu_o}\right) \tag{1}$$

where μ_o and μ_h are shear moduli at the locations of upper and lower surfaces of the layer, respectively, β is the material gradation parameter and, largely to render the incumbent analysis tractable, the spatial variation of Poisson's ratio is assumed to be negligible

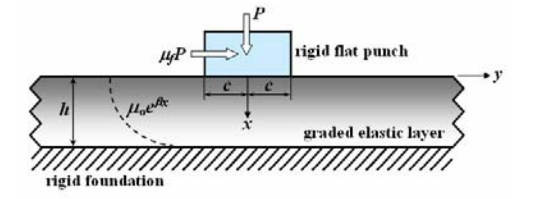

Fig. 1. Problem configuration for a functionally graded layer loaded by a frictional sliding flat punch.

throughout the medium such that ν= constant.

Upon denoting $u(x,y)$ and $v(x,y)$ as the displacement components in the x- and y-directions, respectively, a system of equilibrium equations governing the plane elastic behavior is given by

$$\nabla^2 u+\frac{2}{\kappa-1}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x \partial y}\right)+\frac{\beta}{\kappa-1}\left[(1+\kappa)\frac{\partial u}{\partial x}+(3-\kappa)\frac{\partial v}{\partial y}\right]=0 \qquad (2)$$

$$\nabla^2 v+\frac{2}{\kappa-1}\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 u}{\partial x \partial y}\right)+\beta\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)=0 \qquad (3)$$

and the stress components can be evaluated from the constitutive relations

$$\sigma_{xx}=\frac{\mu_o e^{\beta x}}{\kappa-1}\left[(1+\kappa)\frac{\partial u}{\partial x}+(3-\kappa)\frac{\partial v}{\partial y}\right] \qquad (4)$$

$$\sigma_{yy}=\frac{\mu_o e^{\beta x}}{\kappa-1}\left[(1+\kappa)\frac{\partial v}{\partial y}+(3-\kappa)\frac{\partial u}{\partial x}\right] \qquad (5)$$

$$\tau_{xy}=\mu_o e^{\beta x}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \qquad (6)$$

where $\kappa=3-4\nu$ for the plane strain and $\kappa=(3-\nu)/(1+\nu)$ for the plane stress.

The Fourier integral transform method is employed to solve the above governing field equations so that the general expressions for the displacement and stress components are readily obtained as

$$u=-\frac{i}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{4}F_j\, m_j\, e^{n_j x-isy}\,ds \qquad (7)$$

$$v=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{4}F_j\, e^{n_j x-isy}\,ds \qquad (8)$$

$$\sigma_{xx}=\frac{\mu_o e^{\beta x}}{1-\kappa}\frac{i}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{4}F_j\left[(1+\kappa)m_j n_j+s(3-\kappa)\right]e^{n_j x-isy}\,ds \qquad (9)$$

$$\sigma_{yy}=\frac{\mu_o e^{\beta x}}{1-\kappa}\frac{i}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{4}F_j\left[s(1+\kappa)+(3-\kappa)m_j n_j\right]e^{n_j x-isy}\,ds \qquad (10)$$

$$\tau_{xy}=\frac{\mu_o e^{\beta x}}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{4}F_j\left(n_j-s\,m_j\right)e^{n_j x-isy}\,ds \qquad (11)$$

where s is the transform variable, $i=(-1)^{1/2}$, $F_j(s)$, j=1,..,4, are arbitrary unknowns, $n_j(s)$, j=1,..,4, are the roots of the characteristic equation

$$\left(n^2+\beta n-s^2\right)^2+\left(\frac{3-\kappa}{1+\kappa}\right)\beta^2 s^2=0 \qquad (12)$$

from which it can be shown that

$$n_j=-\frac{\beta}{2}+\sqrt{\frac{\beta^2}{4}+s^2-i(-1)^j\beta s\left(\frac{3-\kappa}{1+\kappa}\right)^{1/2}}\ ;\ \mathrm{Re}(n_j)>0,\ j=1,2 \qquad (13)$$

$$n_j=-\frac{\beta}{2}-\sqrt{\frac{\beta^2}{4}+s^2+i(-1)^j\beta s\left(\frac{3-\kappa}{1+\kappa}\right)^{1/2}}\ ;\ \mathrm{Re}(n_j)<0,\ j=3,4 \qquad (14)$$

and $m_j(s)$, j=1,..,4, are given for each root $n_j(s)$, j=1,..,4, as

$$m_j=\frac{(\kappa-1)(n_j^2+\beta n_j)-(1+\kappa)s^2}{[2n_j+(\kappa-1)\beta]s} \qquad (15)$$

In the contact problem at hand (see Fig. 1), a vertical displacement is imposed *a priori* over the contact region, $|y|<c$, via the prescribed punch profile on the layer upper surface, with the tractions being unknown beneath the punch as

$$u(0,y)=\delta_o,\ \tau_{xy}(0,y)=\mu_f\sigma_{xx}(0,y)\ ;\ |y|<c \qquad (16)$$

while the region outside the contact, $|y|>c$, is traction-free and the layer is in full adhesion with a rigid foundation such that both the vertical and horizontal displacements are zero at its lower surface

$$\sigma_{xx}(0,y)=0,\ \tau_{xy}(0,y)=0\ ;\ |y|>c \qquad (17)$$

$$u(h,y)=0,\ v(h,y)=0\ ;\ |y|<\infty \qquad (18)$$

where δ_0 is the indentation depth. In addition, the equilibrium condition should be satisfied as

$$\int_{-c}^{c} \sigma_{xx}(0,y)\,dy = -P \tag{19}$$

in which P is the resultant contact force.

As can be inferred from the above, the fixed conditions, Eq. (18), can be applied to eliminate the two out of the four unknowns, $F_j(s)$, j=1,..,4, for the elastic field and the mixed conditions in Eqs. (16) and (17) would yield, in principle, a pair of integral equations for the remaining two unknowns.

3. Integral equation for the contact mechanics

The contact pressure distribution is to be determined from the requirement that the displacements developed at the layer upper surface due to the arbitrary unknown normal and tangential tractions in the contact region conform to the punch profile for the complete contact to be maintained. Hence, in deriving the integral equation that relates such surface displacements to the unknown contact pressure distribution, the expressions for the displacements in the graded layer subjected to the fixed condition in Eq. (18) as well as the arbitrary tractions acting on the contact region should be obtained. After some algebraic manipulations, the corresponding displacements can be written in the form as

$$u(0,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[N_{11}(s)\bar{\sigma}(s) + iN_{12}(s)\bar{\tau}(s)\right]e^{-isy}\,ds \tag{20}$$

$$v(0,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[-iN_{21}(s)\bar{\sigma}(s) + N_{22}(s)\bar{\tau}(s)\right]e^{-isy}\,ds \;;\quad |y| < \infty \tag{21}$$

where $\bar{\sigma}(s)$ and $\bar{\tau}(s)$ are the Fourier-transformed normal and tangential tractions, respectively, on the upper surface of the layer

$$\bar{\sigma}(s) = \int_{-c}^{c}\sigma(r)e^{isr}\,dr,\ \bar{\tau}(s) = \int_{-c}^{c}\tau(r)e^{isr}\,dr \tag{22}$$

and the functions $N_{jk}(s)$, j,k=1,2, are dependent on the elastic parameters of the graded medium and the Fourier variable s as well.

For the purpose of extracting the correct nature of singularities the current contact problem may have and excluding the possibility of rigid body displacements, the displacements in Eqs. (20) and (21) are differentiated to yield

$$\frac{\partial u}{\partial y}(0,y) = -\frac{1}{2\pi}\int_{-c}^{c}\left[iK_{11}(y,r)\sigma(r) - K_{12}(y,r)\tau(r)\right]dr \;;\quad |y| < \infty \tag{23}$$

$$\frac{\partial v}{\partial y}(0,y) = -\frac{1}{2\pi}\int_{-c}^{c}\left[K_{21}(y,r)\sigma(r) + iK_{22}(y,r)\tau(r)\right]dr \;;\quad |y| < \infty \tag{24}$$

where the kernel functions, $K_{jk}(y,r)$, j,k=1,2, are given by

$$K_{jk}(y,r) = \int_{-\infty}^{\infty} s\,N_{jk}(s)e^{is(r-y)}\,ds;\ j,k = 1,2 \tag{25}$$

accompanied by the following asymptotic behavior of the integrands as the variable s tends to infinity:

$$\lim_{|s|\to\infty} s\,N_{11}(s) = \lim_{|s|\to\infty} s\,N_{22}(s) = N_1^{\infty}\frac{s}{|s|} \tag{26}$$

$$\lim_{|s|\to\infty} s\,N_{12}(s) = \lim_{|s|\to\infty} s\,N_{21}(s) = N_2^{\infty} \tag{27}$$

in which $N_1^{\infty} = -(\kappa+1)/4\mu_o$ and $N_2^{\infty} = -(\kappa-1)/4\mu_o$.

Upon separating the leading terms from the kernels in Eq. (25) and making use of the Fourier representation of generalized functions [29]

$$\int_0^{\infty}\cos s(r-y)\,ds = \pi\,\delta(r-y) \tag{28}$$

$$\int_0^{\infty}\sin s(r-y)\,ds = \frac{1}{r-y} \tag{29}$$

in which $\delta(r-y)$ is the Dirac delta function, a pair of integral equations is obtained for the unknown tractions $\sigma(y)$ and $\tau(y)$ in the contact region as

$$\frac{\kappa+1}{4\pi\mu_o}\int_{-c}^{c}\frac{\sigma(r)}{r-y}dr + \frac{\kappa-1}{4\mu_o}\tau(y) - \frac{1}{\pi}\int_{-c}^{c}\left[k_{11}(y,r)\sigma(r) + k_{12}(y,r)\tau(r)\right]dr = -\frac{\partial u}{\partial y}(0,y)\;;\quad |y| < c \tag{30}$$

$$\frac{\kappa+1}{4\pi\mu_o}\int_{-c}^{c}\frac{\tau(r)}{r-y}dr - \frac{\kappa-1}{4\mu_o}\sigma(y) + \frac{1}{\pi}\int_{-c}^{c}\left[k_{21}(y,r)\sigma(r) - k_{22}(y,r)\tau(r)\right]dr$$

$$= -\frac{\partial v}{\partial y}(0, y) \; ; \; |y| < c \tag{31}$$

provided the surface slope of the punch profile is prescribed, where $k_{jk}(y,r)$, j,k=1,2, are bounded kernels expressed as

$$k_{ij}(y,r) = \int_0^\infty \left[s N_{ij}(s) - N_1^\infty \right] \sin s(r-y)\, ds \quad ; \quad (i,j) = (1,1),(2,2) \tag{32}$$

$$k_{ij}(y,r) = \int_0^\infty \left[s N_{ij}(s) - N_2^\infty \right] \cos s(r-y)\, ds \quad ; \quad (i,j) = (1,2),(2,1) \tag{33}$$

The fact that the punch slides along the layer upper surface in the presence of friction requires the following relations to be held within the contact region:

$$\sigma_{xx}(0,y) = \sigma(y) = -p(y) \; ; \; |y| < c \tag{34}$$

$$\tau_{xy}(0,y) = \tau(y) = -\mu_f p(y); \; |y| < c \tag{35}$$

where $p(y)$ is the unknown contact pressure. The formulation of the contact problem is thus reduced to solving a Cauchy-type singular integral equation of the second kind for $p(y)$ as

$$\mu_f \frac{\kappa-1}{\kappa+1} p(y) + \frac{1}{\pi}\int_{-c}^{c} \frac{p(r)}{r-y} dr - \frac{1}{\pi}\frac{4\mu_o}{\kappa+1}\int_{-c}^{c} \left[k_{11}(y,r) + \mu_f k_{12}(y,r) \right] p(r)\, dr = f(y) \; ; \; |y| < c \tag{36}$$

in which the function $f(y)$ is given by

$$f(y) = \frac{4\mu_o}{\kappa+1}\frac{\partial u}{\partial y}(0,y) \tag{37}$$

and the contact pressure should be in equilibrium with the resultant contact force P such that

$$\int_{-c}^{c} p(y)\, dy = P \tag{38}$$

4. Solution of the integral equation

Now that the dominant part of the integral equation is solely attributable to the Cauchy singular kernel $1/(r-y)$, the contact pressure $p(y)$ to be sought as the solution to the integral equation can be expressed as [30]

$$p(y) = (c-y)^\chi (c+y)^\omega F(y) \; ; \; |y| < c \tag{39}$$

where χ and ω are the constants to be specified and $F(y)$ is an unknown bounded function.

In the normalized interval, $r=c\eta$ and $y=c\xi$, the nature of the contact pressure is characterized by the fundamental function that corresponds, in this case, to the weight function of Jacobi polynomials [31]. The contact pressure can, therefore, be approximated in terms of a series expansion

$$p(y) = w(\xi)\sum_{n=0}^{\infty} c_n P_n^{(\chi,\omega)}(\xi), \quad w(\xi) = (1-\xi)^\chi (1+\xi)^\omega \; ; \; |\xi| < 1 \tag{40}$$

where c_n, $n \geq 0$, are coefficients to be evaluated, $P_n^{(\chi,\omega)}(\xi)$ are the Jacobi polynomials, and the physics of the flat punch problem dictates that both χ and ω should be negative and determined as

$$\chi = \frac{\theta}{\pi}, \quad \omega = -\frac{\theta}{\pi} - 1, \quad \tan\theta = -\frac{1}{\mu_f}\frac{\kappa+1}{\kappa-1} \quad ; \quad -1 < (\chi,\omega) < 0 \tag{41}$$

from which it is noted that the values of χ and ω as the powers of stress singularity at the leading ($y=c$) and trailing ($y=-c$) edges of the punch, respectively, are functions of only the friction coefficient μ_f and Poisson's ratio ν, as in the case of a homogeneous substrate under frictional contact [9].

With the surface slope of the flat punch being zero within the contact region via Eqs. (16) and (37), after substituting Eq. (40) into Eqs. (36) and (38), truncating the series at $n=N$, and using the properties of the Jacobi polynomials [32], the singular part of the integral equation can be regularized such that the expressions in Eqs. (36) and (38) become

$$\sum_{n=0}^{N} c_n \left[-\frac{1}{2\sin\pi\chi} P_{n-1}^{(-\chi,-\omega)}(\xi) + g_n(\xi) \right] = 0 \tag{42}$$

$$\sum_{n=0}^{N} c_n \int_{-1}^{1} w(\xi) P_n^{(\chi,\omega)}(\xi)\, d\xi = \frac{P}{c} \tag{43}$$

where the function $g_n(\xi)$ is written as

$$g_n(\xi) = -\frac{c}{\pi}\int_{-1}^{1} k_o(\xi,\eta)\, w(\eta) P_n^{(\chi,\omega)}(\eta)\, d\eta \tag{44}$$

$$k_o(\xi,\eta) = \frac{4\mu_o}{\kappa+1}\left[k_{11}(c\xi,c\eta) + \mu_f k_{12}(c\xi,c\eta)\right] \quad (45)$$

To recast the functional equations in (42) and (43) into solvable form, the orthogonality of $P_n^{(\chi,\omega)}(\xi)$ for $\chi+\omega=-1$ is utilized

$$\int_{-1}^{1} w(\xi)P_n^{(\chi,\omega)}(\xi)P_k^{(\chi,\omega)}(\xi)d\xi$$

$$= \begin{cases} 0\ ;\ n \neq k \\ \theta_k^{(\chi,\omega)} = \dfrac{\Gamma(k+\chi+1)\Gamma(k+\omega+1)}{2(k!)^2}\ ;\ n = k \geq 1 \\ \theta_0^{(\chi,\omega)} = \Gamma(\chi+1)\Gamma(\omega+1)\ ;\ n = k = 0 \end{cases} \quad (46)$$

so that a system of linear algebraic equations can be constructed to be solved for c_n, $0 \leq n \leq (N+1)$, as

$$c_0^*\theta_0^{(\chi,\omega)} = 1, \quad -\frac{\theta_k^{(-\chi,-\omega)}}{2\sin\pi\chi}c_{k+1}^* + \sum_{n=0}^{N} d_{kn}c_n^* = 0$$
$$;\ k = 0,1,2,...,N \quad (47)$$

together with the following identities:

$$c_n^* = \frac{c}{P}c_n\ ;\ n = 0,1,2,....,(N+1) \quad (48)$$

$$d_{kn} = \int_{-1}^{1} g_n(\xi)P_k^{(-\chi,-\omega)}(\xi)(1-\xi)^{-\chi}(1+\xi)^{-\omega}d\xi \quad (49)$$

Once the unknown coefficients, c_n, are evaluated from the above system of equations, the contact stress or pressure distribution beneath the flat punch is determined in a straightforward manner as

$$\sigma_{xx}(0,y) = -P(c-y)^{\chi}(c+y)^{\omega}\sum_{n=0}^{N+1}c_n^* P_n^{(\chi,\omega)}\left(\frac{y}{c}\right)$$
$$;\ |y| < c \quad (50)$$

Subsequently, as the additional quantity of primary interest, the expression for the in-plane surface stress component $\sigma_{yy}(0,y)$ on the upper surface of the graded layer can be obtained from the constitutive relations in Eqs. (4) and (5) and Eq. (31) such that

$$\sigma_{yy}(0,y) = -p(y) + \frac{2\mu_f}{\pi}\int_{-c}^{c}\frac{p(r)}{r-y}dr$$
$$+\frac{1}{\pi}\frac{8\mu_o}{\kappa+1}\int_{-c}^{c}\left[k_{21}(y,r) - \mu_f k_{22}(y,r)\right]p(r)dr$$
$$;\ |y| < \infty \quad (51)$$

where in particular, in the normalized interval $r=c\eta$ and $y=c\xi$, the second term on the right-hand side can be evaluated with the aid of the formula [31]

$$L_n(\xi) = \int_{-1}^{+1}\frac{w(\eta)P_n^{(\chi,\omega)}(\eta)}{\eta-\xi}d\eta\ ;\ |\xi| < \infty \quad (52)$$

that has the recurrence relation as

$$L_{n+1}(\xi) = \frac{1}{P_n^{(\chi,\omega)}(\xi)}\left[P_{n+1}^{(\chi,\omega)}(\xi)L_n(\xi) + \frac{2n+1}{n+1}\theta_n^{(\chi,\omega)}\right]$$
$$;\ n \geq 0 \quad (53)$$

$$L_0(\xi) = \frac{\pi}{\sin\pi\chi}\begin{cases}(\xi-1)^{\chi}(\xi+1)^{\omega}\ ;\ |\xi| > 1 \\ (1-\xi)^{\chi}(1+\xi)^{\omega}\cos\pi\chi;\ |\xi| < 1\end{cases} \quad (54)$$

5. Numerical results and discussion

The integral equation in (36) is solved for various combinations of physical parameters (μ_o/μ_h, μ_f, $2c/h$, ν) of the problem under the state of plane strain. A constant Poisson's ratio $\nu=0.3$ is assumed, unless oth-

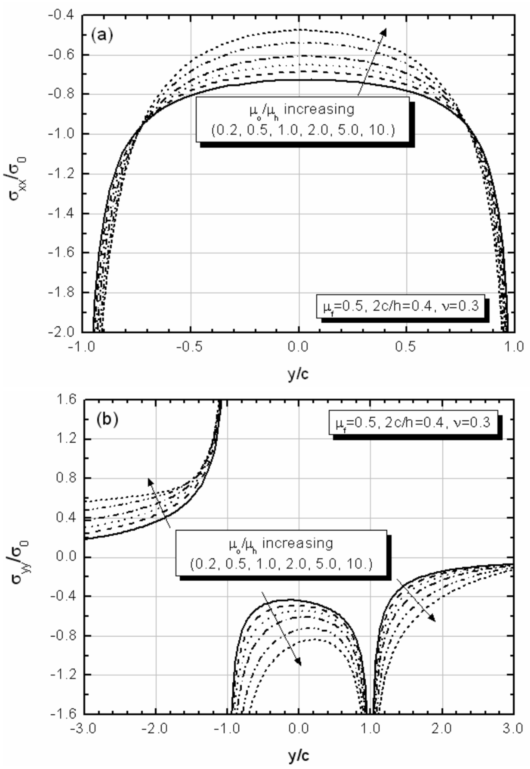

Fig. 2. (a) Distributions of contact stress $\sigma_{xx}(0,y)/\sigma_o$ and (b) in-plane stress $\sigma_{yy}(0,y)/\sigma_o$ on the surface of the graded layer for different values of the shear modulus ratio μ_o/μ_h ($\mu_f=0.5$, $2c/h=0.4$, $\nu=0.3$, $\sigma_o=P/2c$).

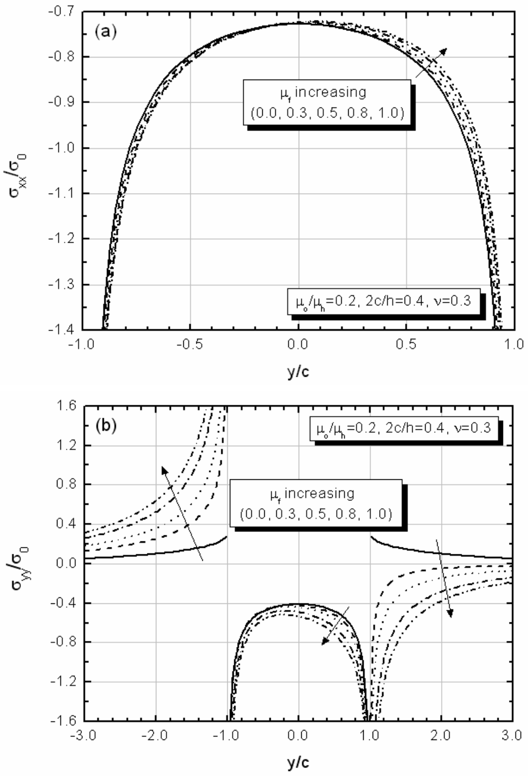

Fig. 3. (a) Distributions of contact stress $\sigma_{xx}(0,y)/\sigma_o$ and (b) in-plane stress $\sigma_{yy}(0,y)/\sigma_o$ on the surface of the graded layer for different values of the friction coefficient μ_f (μ_o/μ_h=0.2, $2c/h$=0.4, ν=0.3, $\sigma_o=P/2c$).

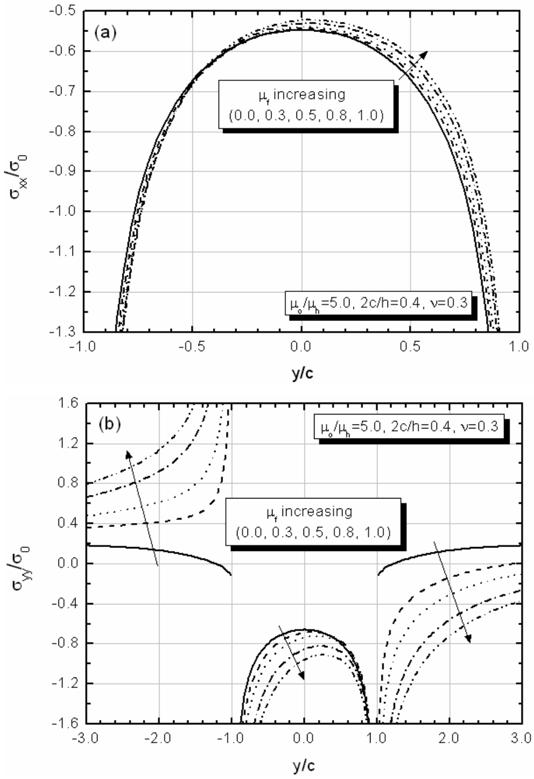

Fig. 4. (a) Distributions of contact stress $\sigma_{xx}(0,y)/\sigma_o$ and (b) in-plane stress $\sigma_{yy}(0,y)/\sigma_o$ on the surface of the graded layer for different values of the friction coefficient μ_f (μ_o/μ_h=5.0, $2c/h$=0.4, ν=0.3, $\sigma_o=P/2c$).

erwise stated. The kernels in Eqs. (32) and (33) as the improper integrals are evaluated employing the Gauss-Legendre quadrature and the integrals in Eqs. (44) and (49) are evaluated on the basis of the Gauss-Jacobi quadrature with sixty collocation points [33], together with a twelve-term expansion of the Jacobi polynomials in Eq. (40).

The distributions of normalized contact pressure $\sigma_{xx}(0,y)/\sigma_o$ and in-plane surface stress component $\sigma_{yy}(0,y)/\sigma_o$ are plotted in Figs. 2a and 2b, respectively, for some values of the shear modulus ratio μ_o/μ_h, where μ_f=0.5, $2c/h$=0.4, and $\sigma_o=P/2c$ is the average contact pressure. In this case, the powers of stress singularity at the trailing ($y=-c$) and leading ($y=c$) edges of the flat punch determined from Eq. (41) are $\omega=-0.5452$ and $\chi=-0.4548$, respectively, which are reflected in Fig. 2a through the greater stress concentrations around the trailing edge of the punch. It can be further inferred from this figure that for the enlarged value of μ_o/μ_h that makes the layer stiffer beneath the punch, there also result in greater stress concentrations around both the trailing and leading edges of the punch, while the magnitude of the contact pressure is reduced around the contact center. The in-plane surface stress as illustrated in Fig. 2(b) clearly depicts that its magnitude is unbounded and discontinuous at both edges of the punch. As the shear modulus ratio increases, it is predictable that the in-plane surface stress behind the trailing edge ($y<-c$) is rendered more tensile, but more compressive in the remaining region of the layer upper surface ($y>-c$), with the relevance that the trailing edge is a more likely location of contact damage, possibly in the form of surface crack initiation and propagation, as was evidenced by the previous experimental observation [8].

The variations of contact stress distributions with the friction coefficient μ_f ranging from 0.0 to 1.0 and $2c/h$=0.4 are next examined in Figs. 3(a) and 3(b) for μ_o/μ_h=0.2 and in Figs. 4(a) and 4(b) for μ_o/μ_h=5.0. As the sliding contact is more frictional, the near-edge behavior in Figs. 3(a) and 4(a) delineates that for the

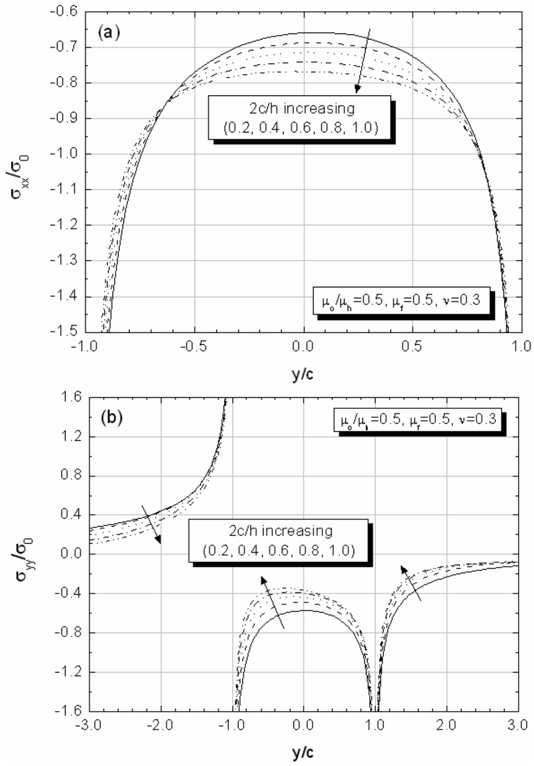

Fig. 5. (a) Distributions of contact stress $\sigma_{xx}(0,y)/\sigma_o$ and (b) in-plane stress $\sigma_{yy}(0,y)/\sigma_o$ on the surface of the graded layer for different values of the flat punch width $2c/h$ (μ_o/μ_h=0.5, μ_f=0.5, ν=0.3, $\sigma_o=P/2c$).

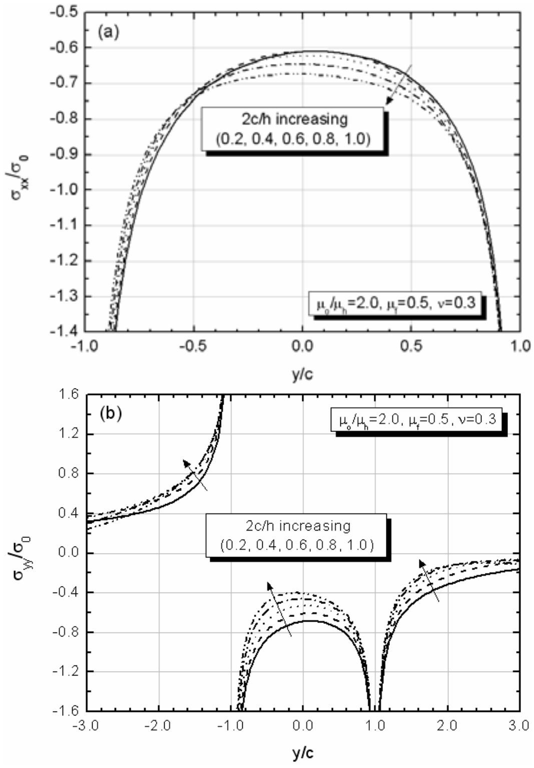

Fig. 6. (a) Distributions of contact stress $\sigma_{xx}(0,y)/\sigma_o$ and (b) in-plane stress $\sigma_{yy}(0,y)/\sigma_o$ on the surface of the graded layer for different values of the flat punch width $2c/h$ (μ_o/μ_h=2.0, μ_f=0.5, ν=0.3, $\sigma_o=P/2c$).

given values of μ_o/μ_h, the stress concentration becomes greater in the vicinity of the trailing edge of the punch, while the stress relaxation is noted near the leading end. This is well correlated with the strengthened and weakened stress singularities that prevail at the trailing and leading edges, respectively, for the increased values of μ_f, as can be estimated from Eq. (41) such that $(\omega, \chi)=(-0.5, -0.5)$ for μ_f=0.0 and $(\omega, \chi)=(-0.5886, -0.4114)$ for μ_f=1.0. The aforementioned trend near the trailing edge in Figs. 3a and 4a is in parallel with that in Figs. 3b and 4b, in the sense that the increase in the friction coefficient may cause the in-plane surface stress to be totally tensile and unbounded behind the trailing edge and to be more compressive ahead of the trailing edge. On the other hand, when the upper side of the layer is stiffer as μ_o/μ_h=5.0, the frictionless punch, μ_f=0.0, is shown to give rise to the in-plane surface stress that is compressive close to both edges of the punch.

Figs. 5 and 6 illustrate the effects of the flat punch width relative to the layer thickness, $2c/h$, on the contact stress field for μ_o/μ_h=0.5 and μ_o/μ_h=2.0, respectively, in which it is assumed that μ_f=0.5. Common features to be remarked of these two material combinations are, as plotted in Figs. 5(a) and 6(a), the increased magnitude of contact pressure within the contact region and the relieved stress concentrations near the edges of the punch for the greater punch width. Such a near-edge response is especially noteworthy around the trailing edge of the punch. The results in Figs. 5(b) and 6(b) show, however, that the punch width $2c/h$ may affect the in-plane component of the surface stress in a different manner, depending on the values of the shear modulus ratio. Specifically, when μ_o/μ_h=0.5, the in-plane surface stress is seen to become less tensile behind the trailing edge for the greater punch width, whereas the reverse trend is observed when μ_o/μ_h= 2.0, with the possible implication that the contacting surface is likely to be less susceptible to the contact-induced surface damage near the trailing edge when the frictional punch of greater width acts on the less stiff side of the graded

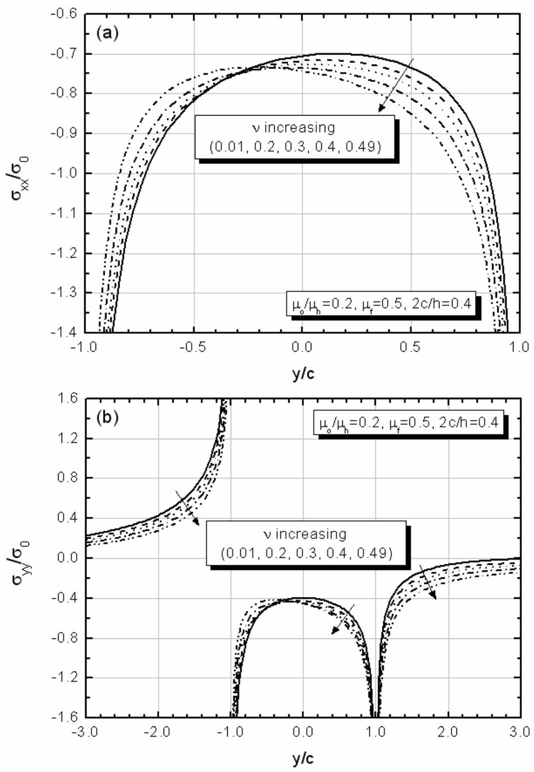

Fig. 7. (a) Distributions of contact stress $\sigma_{xx}(0,y)/\sigma_o$ and (b) in-plane stress $\sigma_{yy}(0,y)/\sigma_o$ on the surface of the graded layer for different values of the Poisson's ratio ν (μ_o/μ_h=0.2, μ_f=0.5, $2c/h$=0.4, $\sigma_o=P/2c$).

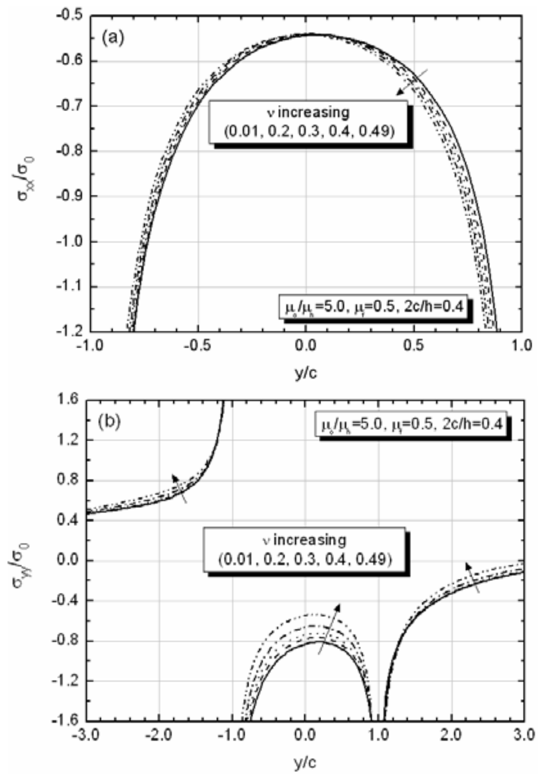

Fig. 8. (a) Distributions of contact stress $\sigma_{xx}(0,y)/\sigma_o$ and (b) in-plane stress $\sigma_{yy}(0,y)/\sigma_o$ on the surface of the graded layer for different values of the Poisson's ratio ν (μ_o/μ_h=5.0, μ_f=0.5, $2c/h$=0.4, $\sigma_o=P/2c$).

layer.

Additional results are provided in Figs. 7 and 8 in order to gain an insight into how the range of Poisson's ratio, $0.01\le\nu\le0.49$, affects the contact stress field for μ_o/μ_h=0.2 and μ_o/μ_h=5.0, respectively. To this end, it is specified that μ_f=0.5 and $2c/h$=0.4. Of interest in this case is the more pronounced influence of Poisson's ratio on the contact stress field for the punch that acts on the less stiff side of the graded layer as μ_o/μ_h=0.2. From the results in Figs. 7a and 8a, it is evident that with the increase in Poisson's ratio, the region near the trailing edge is experiencing relieved stress concentration, while the leading edge region is suffering from the opposite tendency of intensified stress concentration. The strength of stress singularity that decreases at the trailing edge (from ω=−0.5772 at ν=0.01 to ω=−0.5031 at ν=0.49) and increases at the leading edge (from χ=−0.4228 at ν=0.01 to χ=−0.4969 at ν=0.49) with Poisson's ratio supports the foregoing statement. Figure 7b reveals that the in-plane surface stress behind the trailing edge also becomes somewhat alleviated with the increasing Poisson's ratio. The fact that the layer with the greater Poisson's ratio offers diminished restraint to the lateral deformation is deemed to be partly responsible for such stress relaxation, which is particularly true for the graded layer under the frictional sliding contact on its less stiff side. The results in Fig. 8b for μ_o/μ_h=5.0 indicate, however, that the in-plane surface stress behind the trailing edge is affected rather insignificantly by the values of Poisson's ratio.

6. Summary and closure

The problem of frictional contact between a sliding flat punch and a functionally graded layer has been investigated, within the framework of plane elasticity. With the friction coefficient being constant, the lower side of the layer was assumed to be fixed to a rigid foundation. The nonhomogeneity of the graded layer was modeled in terms of an exponential variation of the shear modulus along the thickness direction, while

Poisson's ratio was taken to be constant. The distributions of the contact pressure and the in-plane surface stress component were then obtained for various combinations of material, loading, and geometric parameters of the graded layer under the prescribed contact loading condition. As a result, it was illustrated that when the flat punch acts on the stiffer side of the graded layer or when the punch is more frictional, the stress concentration tends to be intensified especially around the trailing edge of the punch and the in-plane surface stress may be rendered more tensile behind the trailing edge, implying the elevated vulnerability of the layer to the sliding-contact-induced surface damage. On the other hand, the enlarged punch width relative to the layer thickness was shown to relieve the stress concentration near the trailing edge of the punch, although such an effect of the punch width on the in-plane surface stress may differ, depending on the values of the shear modulus ratio. It was further observed that the contact stress field is affected to a larger extent by Poisson's ratio when the graded layer is subjected to the frictional sliding contact on its less stiff side, with the corresponding in-plane surface stress as well as the stress concentration in the trailing edge region being attenuated for the greater values of Poisson's ratio.

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